p-adic Hodge theory: Hodge-Tate and de Rham representations

Ravi Fernando – fernando@berkeley.edu

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1 Setup

Throughout today, K be a p-adic field; namely, a characteristic-0 field that is complete with respect to a fixed discrete valuation, whose residue field k is perfect of characteristic p. For example, K can be any finite extension of $\mathbb{Q}_p$, or the completion of $\mathbb{Q}_p^{ur}$. It cannot be $\overline{\mathbb{Q}}_p$ or $\mathbb{C}_p$, as these are not discretely valued (and the former is not complete).

Set $G_K = \mathrm{Gal}(\overline{K}/K)$ and $\mathbb{C}_K = \widehat{\overline{K}}$. We are interested in studying the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of finite-dimensional $\mathbb{Q}_p$-representations of G_K, but we will soon tensor these up to (semilinear) $\mathbb{C}_K$-representations of G_K. Unless otherwise specified, all representations, Galois cohomology, etc. that we discuss today will be implicitly continuous.

Let $\mathbb{Z}_p(1) = \lim_{\leftarrow n} \mu_{p^n}(\overline{K})$, which is a free $\mathbb{Z}_p$-module of rank 1 equipped with a natural G_K-action. If we fix a basis of $\mathbb{Z}_p(1)$, we can view it as $\mathbb{Z}_p$ with a character $\chi : G_K \to \mathbb{Z}_p^\times$; χ is called the cyclotomic character. Then we let $\mathbb{Z}_p(r) = \mathbb{Z}_p(1)^{\otimes r}$, $\mathbb{Q}_p(r) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$, and so on.

2 $\mathbb{C}_K$-representations

The $\mathbb{Q}_p$-representations of G_K we care about, most notably those coming from étale cohomology, are very difficult to understand. But as Brinon and Conrad explain, "they become *much simpler* after we apply the drastic operation $V \mapsto \mathbb{C}_K \otimes_{\mathbb{Q}_p} V$", where G_K acts on both sides of the tensor product. Note in particular that this G_K-action is *not* $\mathbb{C}_K$-linear; rather, it is $\mathbb{C}_K$-semilinear, in the sense that $g(cv) = g(c)g(v)$ for $c \in \mathbb{C}_K$ and $v \in V$.

Definition 2.1. *A $\mathbb{C}_K$-representation of G_K is a finite-dimensional $\mathbb{C}_K$-vector space W equipped with a continuous G_K-action such that $g(cw) = g(c)g(w)$ for all $c \in \mathbb{C}_K$ and $w \in W$. The*

*Notes for a talk given in a reading course on Galois representations at Berkeley, supervised by Sug Woo Shin. Main reference: Brinon and Conrad, *CMI summer school notes on p-adic Hodge theory*, chapters 2 and 4-6.

category of $\mathbb{C}_K$-representations of G_K (with $\mathbb{C}_K$-linear G_K-equivariant morphisms) is called $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

Facts: $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is an abelian category with reasonable notions of $\oplus$, $\otimes$, duals, and exactness. (Be careful when dealing with semilinearity.) Extension of scalars gives an exact functor $\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

Theorem 2.2. *(Faltings) If K is a p-adic field and X is a smooth proper K-scheme, there is a canonical isomorphism*

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \equiv \bigoplus_q \left(\mathbb{C}_K(-q) \otimes_K H^{n-q}(X, \Omega^q_{X/K}) \right). \tag{1}$$

(The Hodge cohomology groups on the right are just K-vector spaces; they do not have a Galois action. So we could rewrite the right-hand side slightly less canonically as $\bigoplus_q \mathbb{C}_K(-q)^{h^{n-q,q}}$.) This is closely analogous to the classical Hodge decomposition for the de Rham cohomology of compact Kähler manifolds. It tells us in particular that the p-adic étale cohomology of X recovers its Hodge numbers, and the Hodge numbers recover the étale cohomology but only after base-changing up to $\mathbb{C}_K$. Also note that you really do need to tensor up to $\mathbb{C}_K$ in Faltings' theorem, not just $\overline{K}$. (Brinon and Conrad's Example 2.2.4 justifies this precisely.)

Theorem 2.3. *(Tate-Sen) For $r \in \mathbb{Z}$, the G_K-invariants (equivalently, zeroth continuous Galois cohomology) of $\mathbb{C}_K(r)$ are K if $r = 0$ and 0 otherwise. Similarly, $H^1_{\mathrm{cont}}(G_K, \mathbb{C}_K(r))$ is trivial if $r \neq 0$ and 1-dimensional over K if $r = 0$.*

There exists a version of this for much more general characters η replacing χ^r, but we won't be concerned with this. Note that the first part of this statement says concretely that $\mathbb{C}_K$ has no transcendental G_K-invariants, and it has no nonzero elements on which G_K acts by the character χ^{-r} for $r \neq 0$. One reason we care about (continuous) Galois cohomology is that $H^1_{\mathrm{cont}}(G_K, W)$ classifies extensions $0 \to W \to W' \to \mathbb{C}_K \to 0$.

3 Hodge-Tate representations

The Hodge-Tate property is the first and weakest "niceness" property in p-adic Hodge theory, followed by de Rham, semistable, and crystalline. Let's start with a preliminary definition:

Definition 3.1. *A representation W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is* Hodge-Tate *if W is isomorphic to a (necessarily finite) direct sum of representations of the form $\mathbb{C}_K(-q)$, for some choices of q. We say an object in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is* Hodge-Tate *if its base change to $\mathbb{C}_K$ is. In either case, the* Hodge-Tate numbers *are the numbers q that appear in the decomposition, with their respective multiplicities.*

This definition is a little dissatisfying, because it relies on a non-canonical decomposition. (For example, it is not immediately obvious from the definition that Hodge-Tate weights are well-defined.) Faltings' theorem is a nicer statement: it says that the $\mathbb{C}_K$-representations of G_K coming from geometry can be identified *canonically* as $W \equiv \oplus_q C_K(-q) \otimes_K W\{q\}$ for some

K-vector spaces $W\{q\}$. This implies that W is Hodge-Tate with the multiplicity of each Hodge-Tate weight q equal to the Hodge number $h^{n-q,q}$, but it's better, since this equality comes from a canonical (not just abstract) isomorphism of vector spaces. We would like to redefine Hodge-Tate representations in a way that identifies these $W\{q\}$ canonically.

Recall from the theorem of Tate-Sen that the ring of invariants $\mathbb{C}_K^{G_K}$ is just K, and that $\mathbb{C}_K(-q)^{G_K}$ is 0 for $q \neq 0$. This allows us to canonically identify the "$\mathbb{C}_K(-q)$-piece" of a $\mathbb{C}_K$-representation for each q: let $W\{q\} = W(q)^{G_K}$. Then if W is Hodge-Tate, $W\{q\}$ will be a K-vector space whose dimension equals the multiplicity of the Hodge-Tate weight q. Moreover, $W\{q\}$ makes sense for representations that are not Hodge-Tate, which allows us to redefine Hodge-Tate as follows.

Definitions and Lemma 3.2. *For any $\mathbb{C}_K$-representation of G_K, let $\xi_W : \bigoplus_q \mathbb{C}_K(-q) \otimes_K W\{q\} \to W$ be the map built from the pieces*

$$\mathbb{C}_K(-q) \otimes_K W\{q\} \hookrightarrow \mathbb{C}_K(-q) \otimes_K W(q) \twoheadrightarrow \mathbb{C}_K(-q) \otimes_{\mathbb{C}_K} W(q) = W \tag{2}$$

This is a morphism in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$, and it is always an injection. (This is a lemma of Serre and Tate, proved by infinite descent on the number of simple tensors in a counterexample.) We redefine a Hodge-Tate representation to be any object W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ such that ξ_W is an isomorphism. The other definitions above proceed similarly; for example, the multiplicity of a Hodge-Tate weight q is the K-dimension of $W\{q\}$.

We now consider a category that we will show is equivalent to the category of Hodge-Tate $\mathbb{C}_K$-representations of G_K:

Definition 3.3. *Let $\mathrm{Gr}_{K,f}$ be the abelian category of finite-dimensional graded vector spaces over K, with morphisms preserving degrees. We define the functor $\underline{D} = \underline{D}_{HT}$ from $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ to $\mathrm{Gr}_{K,f}$ by $\underline{D}(W) = (B_{HT} \otimes_{\mathbb{C}_K} W)^{G_K}$, where the grading on the output comes from the grading on B_{HT}.*

Example: if W is the Hodge-Tate representation $\mathbb{C}_K(1) \oplus \mathbb{C}_K(2)^{\oplus 2}$, then we have

$$B_{HT} \otimes W = \bigoplus_{q \in \mathbb{Z}} (\mathbb{C}_K(q+1) \oplus \mathbb{C}_K(q+2)^{\oplus 2}), \tag{3}$$

so $\underline{D}(W) = K\langle -1 \rangle \oplus K^{\oplus 2}\langle -2 \rangle$; that is, it has a copy of K in the (-1)st graded piece and a copy of $K^{\oplus 2}$ in the (-2)nd graded piece.

One observes easily that $\underline{D}$ is left-exact, but it is not right-exact in general. However, a cute argument shows the following.

Proposition 3.4. *Given a short exact sequence $0 \to W' \to W \to W'' \to 0$, if W is Hodge-Tate, then W' and W'' are. (The converse is false.) Moreover, applying $\underline{D}$ to such a short exact sequence yields a short exact sequence in $\mathrm{Gr}_{K,f}$.*

Proposition 3.5. *The Hodge-Tate property is insensitive to finite and inertial extensions. Namely, for W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and K'/K finite, W is Hodge-Tate if and only if it is Hodge-Tate in $\mathrm{Rep}_{\mathbb{C}_K}(G_{K'})$, if and only if it is Hodge-Tate in $\mathrm{Rep}_{\mathbb{C}_K}(I_K)$, where $I_K = G_{\widehat{K^{ur}}}$.*

Now we will introduce a ring that will allow us to repackage much of the preceding discussion into a nice equivalence of categories.

Definition 3.6. *Let B_{HT} be the ring $\bigoplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)$, with multiplication sending $\mathbb{C}_K(q) \times \mathbb{C}_K(q')$ to $\mathbb{C}_K(q+q')$. This has several types of structure: $\mathbb{C}_K$-vector space, grading, G_K-action, ring structure, (topology—do we still care about this?), and they are compatible in the expected ways. Note that if we choose a (non-canonical) generator t of $\mathbb{Z}_p(1)$, then $B_{HT} = \mathbb{C}_K[t^{\pm 1}]$.*

Now we construct a functor $\underline{V}$ from $\mathrm{Gr}_{K,f}$ back to $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ that will be a (sometimes) quasi-inverse to $\underline{D}$.

Definition 3.7. *For D in $\mathrm{Gr}_{K,f}$, let $\underline{V}(D) = \mathrm{gr}^0(B_{HT} \otimes_K D)$, where the grading is by the sum of the gradings on the two tensor factors.*

In our example from earlier, $D = K\langle -1\rangle \oplus K^{\oplus 2}\langle -2\rangle$, we get

$$B_{HT} \otimes_K D = \bigoplus_{q\in\mathbb{Z}} \left(\mathbb{C}_K(q)\langle q-1\rangle \oplus \mathbb{C}_K(q)^{\oplus 2}\langle q-2\rangle\right), \tag{4}$$

so $\underline{V}(D) = \mathrm{gr}^0(B_{HT} \otimes_K D) = \mathbb{C}_K(1) \oplus \mathbb{C}_K(2)^{\oplus 2}$. This is great: we recovered an entire $\mathbb{C}_K$-representation from a finite-dimensional graded K-vector space. This makes the following theorem very unsurprising.

Definition 3.8. *The functors $\underline{D}$ and $\underline{V}$ given an equivalence of categories between the Hodge-Tate representations in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and $\mathrm{Gr}_{K,f}$, which behaves well under tensor products and duals.*

In fact, the isomorphism $\underline{V}(\underline{D}(W)) \to W$ for W Hodge-Tate can be realized as the zeroth graded component of the following *comparison morphism*:

$$\gamma_W : B_{HT} \otimes_K \underline{D}(W) \to B_{HT} \otimes_K (B_{HT} \otimes_{\mathbb{C}_K} W) \to B_{HT} \otimes_{\mathbb{C}_K} W. \tag{5}$$

(Here we use the multiplication map $B_{HT} \otimes_K B_{HT} \to B_{HT}$.) Note that this morphism makes sense for any W. It is always a Galois-equivariant injection, respecting gradings, and it is an isomorphism if and only if W is Hodge-Tate.

Note that the preceding equivalence of categories is only valid when the source consists of $\mathbb{C}_K$-representations, not $\mathbb{Q}_p$-representations. The corresponding functor from $\mathrm{Rep}_{HT}(G_K)$ (Hodge-Tate representations over $\mathbb{Q}_p$) to $\mathrm{Gr}_{K,f}$ is not fully faithful, because tensoring up to $\mathbb{C}_K$ loses a lot of finer information. For example, if η is a character with finite image, then $(B_{HT} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta))^{G_K}$ is isomorphic to $(B_{HT} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p)^{G_K} = K\langle 0\rangle$. So we want to construct a ring, some functors, and a class of nice representations that will be finer than what we have done so far. These will necessary depend on the structure of a $\mathbb{Q}_p$-representation, rather than a $\mathbb{C}_K$-representation.

4 Period ring formalism

Many of the constructions of the previous section work in much greater generality, and it is useful to record them as a formalism that can then be applied for de Rham, semistable, and crystalline representations.

Let F be a field, G a group, and B an F-algebra that is a domain, equipped with a G-action that respects its F-algebra structure. (Think of F as $\mathbb{Q}_p$, G as G_K, and B as B_{HT}.) Assume that the invariant subalgebra $E = B^G$ is a field (e.g. K). Let $C = \operatorname{Frac} B$, which naturally gets a G-action. Note that none of these objects are assumed to have a topology.

Definition 4.1. *We say B is* (F,G)-regular *if $C^G = B^G$ and if every nonzero $b \in B$ whose F-span is G-stable is a unit in B.*

Example: if $F = \mathbb{Q}_p$, $G = G_K$, and $B = B_{HT}$ as above, then one can check that the only invariants in $C = \operatorname{Frac} B_{HT}$ are the "scalars" $K\langle 0\rangle$, and the only $\mathbb{Q}_p$-lines in B_{HT} preserved by G_K are the direct summands $\mathbb{C}_K(q)$ of B_{HT}, which are spanned by units.

For each (F,G)-regular ring B, we get a comparison morphism and a corresponding class of "nice" representations, as follows.

Definition 4.2. *If B is (F,G)-regular and V is a finite-dimensional F-representation of G, set $D_B(V) = (B \otimes_F V)^G$, and consider the comparison morphism*

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_E (B \otimes_F V) = (B \otimes_E B) \otimes_F V \to B \otimes_F V. \tag{6}$$

*This is always a B-linear G-equivariant injection, and we say that V is B-*admissible *if α_V is an isomorphism.*

Theorem 4.3. *If $\operatorname{Rep}_F^B(G)$ denotes the full subcategory of B-admissible representations, then the functor $D_B : \operatorname{Rep}_F^B(G) \to \operatorname{Vec}_E$ (the category of finite-dimensional E-vector spaces) is exact and faithful, and B-admissible representations are closed under subquotients, tensor products, and duals. Moreover, D_B behaves well under tensor products and duals.*

5 de Rham representations

The Hodge-Tate property is a good first step towards measuring niceness of p-adic representations, but we want to develop some stronger conditions, capable of detecting finer data. We will look to improve it by upgrading from graded vector spaces to filtered vector spaces.

A de Rham representation in $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$ (not $\operatorname{Rep}_{\mathbb{C}_K}(G_K)$) will be a B_{dR}-admissible representation, where B_{dR} is a field to be defined. Once we define it, it will be true that de Rham representations are Hodge-Tate. Moreover, B_{dR} will come with a filtration; taking associated graded objects will turn B_{dR} into B_{HT}, and will turn $D_{dR}(V)$ into $D_{HT}(V)$ when V is de Rham.

Let's construct B_{dR}. We start with some objects that came up in my perfectoid space talk last month. Let $\mathcal{O}_{\mathbb{C}_K}$ be the ring of integers of $\mathbb{C}_K$, and consider the $\mathbb{F}_p$-algebra $\mathcal{O}_{\mathbb{C}_K}/(p)$. This isn't a very nice ring, since it has tons of nilpotents, but we can make it nicer by taking its inverse limit along the Frobenius map: $\mathcal{O}_{\mathbb{C}_K}^\flat = \lim_{\leftarrow \varphi} \mathcal{O}_{\mathbb{C}_K}/(p)$. This is the tilt, in the sense of perfectoid spaces; $\mathcal{O}_{\mathbb{C}_K}^\flat$ is a perfect $\mathbb{F}_p$-algebra and an integral domain, and its fraction field is algebraically closed by the tilting equivalence.

Next, we take the Witt vectors $W(\mathcal{O}_{\mathbb{C}_K}^\flat)$. This produces the p-adic period ring A_{inf}. We then get a map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$, defined either by an explicit formula or by some adjunction property, that fits into a commutative square

$$\begin{array}{ccc} A_{\mathrm{inf}} & \xrightarrow{\theta} & \mathcal{O}_{\mathbb{C}_K} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{C}_K}^\flat & \longrightarrow & \mathcal{O}_{\mathbb{C}_K}/(p) \end{array}$$

where the bottom map is the projection onto the last term in the inverse limit, and the vertical maps are modding out by p. Note that $p = 0$ in the bottom row here, but p is merely topologically nilpotent in the top row. We then adjoint $1/p$ to the top row, yielding a map $\theta_{\mathbb{Q}} : A_{\mathrm{inf}}[1/p] \to \mathcal{O}_{\mathbb{C}_K}[1/p] = \mathbb{C}_K$.

It turns out that $\ker \theta_{\mathbb{Q}}$ (and ever $\ker \theta$) is a principal ideal generated by an element $\xi = p - [\underline{p}]$. (Here, $\underline{p}$ is the element $(\dots, p^{1/p^2}, p^{1/p}, p) \in \mathcal{O}_{\mathbb{C}_K}^\flat = \lim_{\leftarrow \varphi} \mathcal{O}_{\mathbb{C}_K}/(p)$, for some fixed choice of compatible p-power roots, and $[\underline{p}]$ is its Teichmüller lift to A_{inf}.) We set B_{dR}^+ to be the $(\ker \theta_{\mathbb{Q}})$-adic completion of $A_{\mathrm{inf}}[1/p]$, and $B_{dR} = \operatorname{Frac} B_{dR}^+$.

Some things that can be checked: B_{dR}^+ is a complete DVR with residue field $\mathbb{C}_K$, where the maximal ideal is $B_{dR}^+ \cdot \ker \theta_{\mathbb{Q}}$. This gives an exhaustive and separated filtration of B_{dR} by filtered pieces $\mathfrak{m}^n B_{dR}^+$, for $n \in \mathbb{Z}$. Both B_{dR}^+ and B_{dR} have natural Galois actions by tracing through each step of the construction, and the Galois action respects the filtration. The associated graded algebra of B_{dR} is isomorphic to B_{HT}, compatibly with Galois action.